Math 122 Wednesday, October 12

V over a field F $\quad \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}$ $\quad a \neq 0 \Rightarrow a^{-1}$ exists

group under + $\quad v+w, 0, -v$

scalar mult $\quad c \cdot v$

ex: $V = F$, $V = F^n = \{v = (c_1, \dots, c_n)\}$ $n \geq 1$

$V = C[0,1] = \{f : [0,1] \to \mathbb{R}, \text{continuous}\}$ a topological space

use continuous linear maps

$S = \{v_1, \dots, v_n\} \subset V$ Then $\text{Span}\{v_i\} = W \subset V$ the smallest subspace of V containing v_i.

Let $T : F^n \to V$ be $T(c_1, \dots, c_n) = \sum_{i=1}^{n} c_i v_i = c_1 v_1 + \dots + c_n v_n$

Then $\text{Span}\, S = \text{Im}\, T$.

If $\text{Im}\, T = V$ is surjective we say that S spans V.

If T is injective, i.e. $\sum c_i v_i = 0$ iff each $c_i = 0$, then S is linearly independent.

S is a basis if it is linearly independent and spans, i.e., if T is an isomorphism.

defn F is finite dimensional if there is a finite spanning set. $\quad F^n \underset{T}{\twoheadrightarrow} V$

Prop If S is a finite set spanning V then some subset of S forms a basis.

Pf: If S is linearly independent $\Rightarrow$ done. If not, have a linear combination $\sum_{i=1}^{n} c_i v_i = 0$ some $c_i \neq 0$. Assume $c_n \neq 0$. Then $(-c_n)^{-1}(c_1 v_1 + \dots + c_{n-1} v_{n-1}) = v_n$ So $S' = S - \{v_n\}$ spans. If independent, stop. If not, continue this process.

Prop If $L = \{v_i\}$ is a set of linearly independent vectors and V is finite dimensional then we can complete L to a basis.

Pf: If L spans, we are done. If not, let S be a finite spanning set. Take $v \in S$ which is not in $W = \text{Span}(L)$ (if this does not exist then $S \subset W$ and L spans). Then $L \cup \{v\}$ is linearly independent. If not $\sum_{i=1}^{n} c_i v_i + cv = 0$. If $c = 0$ then all $c_i = 0$ because L is linearly independent. So $c \neq 0$ and $v = \sum_{i=1}^{n} (-c)^{-1} c_i v_i \Rightarrow v \in \text{Span}(L)$.

So $L' = L \cup \{v\}$ is linearly independent. If spans, stop. If not, continue.

As S is finite and spanning this process terminates.

<u>Thm</u> Let $L, S \subset V$ such that L is linearly independent and S spans. Then $\underset{n}{\#L} \le \underset{m}{\#S}$

Pf: Assume that $n > m$ and derive a contradiction to the linear independence of L. For each $w_j \in L$ write $w_j = \sum_{i=1}^{m} a_{ij} v_i$ where $S = \{v_1, \dots, v_m\}$. Can do this because S spans. Then:

$$\sum_{j=1}^{n} c_j w_j = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} a_{ij} w_i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} c_j a_{ij}\right) w_i .$$

We can make $\sum c_j w_j = 0$ without all of the $c_j = 0$ by making all the $\sum_{j=1}^{m} c_j a_{ij} = 0$. To do this solve the system of m linear equations $\sum_{i=1}^{n} a_{ij} X_i = 0$ in n unknowns X_i. As $n > m$ there are more variables than homogeneous linear equations so we have a non-zero solution.

<u>Cor</u> Any finite dimensional V has a basis and all bases of V have the same number of elements $n = \dim V \ge 0$.

Pf: Existence by reducing a spanning set. If B, B' are bases since B spans and B' linearly independent $\#B \ge \#B'$. Similarly $\#B \le \#B' \Rightarrow \#B = \#B'$

Note: $F^n \ncong F^m$ for $n \ne m$. Note: the choice of basis is <u>not</u> unique

Even in $V = F^n$, which has a fixed basis, you may want a different basis to analyse a linear transformation T.

Note if $\{f_i\}$ form a basis for W then every $w \in W$ is s.t. $w = \sum e_i f_i$ <u>uniquely</u>. If $w = \sum d_i f_i$ then $\sum (e_i - d_i) f_i = 0$.

Let $T: V \to W$ where $\{e_i\}$ is a basis for V and $\{f_i\}$ is a basis for W.
$T(e_j) = \sum c_{ij} f_i$. The matrix for T is $A = (c_{ij})$; it's jth column is $T(e_j)$ expressed in terms of the f_i.

Generally

$$\begin{tikzcd} V \arrow[r, "T"] & W \\ F^n \arrow[u, "\wr", "\text{basis } e"'] \arrow[r, "\text{left mult by } A"'] & F^m \arrow[u, "\wr"', "\text{basis } f"] \end{tikzcd}$$

ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ wrt e_1, e_2.

$e_1 \mapsto e_1$ $\quad$ Let $e_1' = e_1$, $e_2' = e_1 + e_2$

$e_2 \mapsto 2e_2$

$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ wrt e_1, e_2.